

Deformed Harmonic Oscillator Algebras defined by their Bargmann representations.

Michèle IRAC-ASTAUD and Guy RIDEAU
Laboratoire de Physique Théorique de la matière condensée
Université Paris VII
2 place Jussieu F-75251 Paris Cedex 05, FRANCE

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Abstract

Deformed Harmonic Oscillator Algebras are generated by four operators two mutually adjoint a and a^\dagger , and two self-adjoint N and the unity 1 such as :

$$[a, N] = a, \quad [a^\dagger, N] = -a^\dagger, \quad a^\dagger a = \psi(N) \quad \text{and} \quad aa^\dagger = \psi(N+1).$$

The Bargmann Hilbert space is defined as a space of functions, holomorphic in a ring of the complex plane, equipped with a scalar product involving a true integral. In a Bargmann representation, the operators of a Deformed Harmonic Oscillator Algebra act on a Bargmann Hilbert space and the creation (or the annihilation operator) is the multiplication by z . We discuss the conditions of existence of Deformed Harmonic Oscillator Algebras assumed to admit a given Bargmann representation.

1 Introduction

In previous papers [1], [2], [3], we introduced what we have called Deformed Harmonic Oscillator Algebras (DHOA).

Definition 1.1 *A Deformed Harmonic Oscillator Algebra is a free algebra generated by four operators :*

- the annihilation operator a , the creation operator a^\dagger that are mutually adjoint,
- the self-adjoint energy operator N and the unity 1

satisfying the following commutation relations :

$$[a, N] = a, \quad [a^\dagger, N] = -a^\dagger, \quad a^\dagger a = \psi(N), \quad aa^\dagger = \psi(N + 1) \quad (1)$$

where ψ is a real analytical function.

When $\psi(N) = N + \lambda$, λ being in the field, we recover the commutation relations of the usual harmonic oscillator algebra.

Generalizing the pioneer work of Bargmann [12] for the usual harmonic oscillator, we have studied in [1], [2] and [3] the Bargmann representations of the DHOA, defined by (1).

Definition 1.2 *A Bargmann Hilbert space is a space \mathcal{S} of functions, holomorphic on a ring D of the complex plane, the scalar product of which is written with a true integral on the form :*

$$(g, f) = \int F(z\bar{z}) f(z) \overline{g(z)} dz d\bar{z} \quad (2)$$

A Bargmann representation of a Deformed Harmonic Oscillator Algebra is a representation on a Bargmann Hilbert space such as the annihilation or the creation operator admits eigenvectors generating \mathcal{S} .

Let us stress that in this definition, we discard the occurrence of q -integration in the scalar product, contrarily to many authors, see in particular [8][9].

In this paper, starting from the reciprocal point of view of that developed in [1], [2] and [3], we construct the DHOA defined by a given Bargmann representation. That is, a Bargmann Hilbert space \mathcal{S} being given, we look for DHOA that can be represented by operators acting on \mathcal{S} .

In section 2, we recall briefly the irreducible representations of the DHOA on the basis of the eigenvectors of N and we discuss the existence of coherent states, defined as the eigenstates of the operators a (or a^\dagger). In section 3, we summarize the study of Bargmann representations for the DHOA, done in [1], [2] and [3] and set up the relations between the weight function F defining the scalar product and the function ψ characterizing the DHOA. Section 4

is devoted to the true subject of this paper. We study the conditions of existence of a function ψ and consequently of a DHOA defined by (1) when we impose that the DHOA has a representation on a given Bargmann Hilbert space. We prove that if the weight function F fulfills sufficient and necessary conditions, the construction can be performed. Furthermore, we obtain a necessary condition on the function ψ in order that the DHOA admits a Bargmann representation when the domain of the coherent states is a true ring. In section 5, we treat several examples illustrating our construction, showing in particular that the sufficient conditions obtained in Section 4, are not necessary.

2 Representations

2.1 Eigenvectors of N

Let $|0\rangle$ be the eigenvector of N with eigenvalue μ that is assumed to be zero to simplify the notations and is restored when necessary for discussions. We built the normalized vectors $|n\rangle$

$$|n\rangle = \begin{cases} \lambda_n a^{\dagger n} |0\rangle, & n \in N^+ \\ \lambda_n a^{-n} |0\rangle, & n \in N^- \end{cases} \quad (3)$$

with

$$\lambda_n^{-2} = \psi(n)! = \begin{cases} 1, & n = 0 \\ \prod_{i=1}^n \psi(i), & n \in N^+ - \{0\} \\ \prod_{i=0}^{n+1} \psi(i), & n \in N^- \end{cases} \quad (4)$$

N^+ and N^- are the set of integers ≥ 0 and < 0 .

The vectors $|n\rangle$ are the eigenvectors of N with eigenvalue n and span the Hilbert space \mathcal{H} . The condition that $\langle n | aa^\dagger | n \rangle$ is strictly positive, restricts the spectrum of N . The elements of SpN are the integers in any interval $[\nu, \nu']$ in which the function ψ is finite and strictly positive. Eventually one of the edge or both can be infinity. When one edge is finite, it is an integer when $\mu = 0$, and a zero of ψ . We thus get different types of representations [6], [7], [10], [11] according as ψ has no zero, one zero or more, the distance between two consecutive zeros being integer. The representations are defined by :

$$\begin{cases} a^\dagger | n \rangle = (\psi(n+1))^{1/2} | n+1 \rangle \\ a | n \rangle = (\psi(n))^{1/2} | n-1 \rangle, \quad n \in SpN \\ N | n \rangle = (n) | n \rangle \end{cases} \quad (5)$$

When μ is kept different from zero, all the relations of this section remain valid provided we change $\psi(\rho)$ into $\psi(\mu + \rho)$.

Proposition 2.1 *By construction, the eigenvalue μ of the starting state $| 0 \rangle$ belongs always to the spectrum of N .*

The first step to build a Bargmann representation requires to study the coherent vectors that constitute the basis vectors of the representation.

2.2 Coherent states

We call coherent states, the eigenvectors of the operator a or a^\dagger .

The state $| z \rangle = \sum_p c_p | p \rangle$ is an eigenvector of the annihilation operator a if the coefficients c_p verify the recursive relation

$$z c_p = \psi(p+1)^{1/2} c_{p+1} \quad (6)$$

When the spectrum of N is finite, we have proved that a and a^\dagger have no eigenvectors, hence no Bargmann representation exists.

When the spectrum of N is not upper bounded, let us denote this spectrum $SpN = \lambda + N^+$, with the convention $\lambda + N^+ = \mathbb{Z}$ when $\lambda = -\infty$. It results from Proposition 2.1, that $\lambda \in N^- + \{0\}$. The eigenvectors $| z \rangle$ of a take the form :

$$| z \rangle = \sum_{n=-1}^{\lambda} z^n (\psi(n)!)^{1/2} | n \rangle + \sum_{n=0}^{\infty} z^n (\psi(n)!)^{-1/2} | n \rangle \quad (7)$$

The domain D of existence of the coherent states depends on the function ψ . Indeed, $| z \rangle$ belongs to the Hilbert space spanned by the basis $| n \rangle$ only if the series in the right hand side of (7) are convergent in norm, see the detailed discussion in [1], [2], [3].

An analogous reasoning holds when SpN is not lower bounded, for the eigenstates of a^\dagger , it results from Proposition 2.1 that, in this case, $\lambda \in N^+ - \{0\}$. We have proved that the eigenstates of a and a^\dagger never coexist. Let us summarize the results :

Proposition 2.2 *The eigenvectors of the annihilation operator of a Deformed Harmonic Oscillator Algebra exist if the function ψ occurring in the relations (1) belongs to two classes :*

- ψ is a strictly positive function without singularity on the whole real axis, $SpN = Z$, and

$\lim_{p \rightarrow -\infty} \psi(p)^{1/2} < \lim_{p \rightarrow \infty} \psi(p)^{1/2}$, then the domain of existence of the coherent states is :

$$D = \{z; \lim_{p \rightarrow -\infty} \psi(p)^{1/2} < |z| < \lim_{p \rightarrow \infty} \psi(p)^{1/2}\}.$$

- It exists $\lambda \in N^- + \{0\}$ such that $\psi(\lambda) = 0$ and ψ is strictly positive without singularity when $x > \lambda$, $SpN = \lambda + N^+$ then :

$$D = \{z; |z| < \lim_{p \rightarrow \infty} \psi(p)^{1/2}\}.$$

and

Proposition 2.3 *The eigenvectors of the creation operator of a Deformed Harmonic Oscillator Algebra exist if the function ψ occurring in the relations (1) belongs to two classes :*

- ψ is a strictly positive function without singularity on the whole real axis, $SpN = Z$, and

$$\lim_{p \rightarrow -\infty} \psi(p)^{1/2} > \lim_{p \rightarrow \infty} \psi(p)^{1/2},$$

then the domain of existence of the coherent states is

$$D = \{z; \lim_{p \rightarrow \infty} \psi(p)^{1/2} < |z| < \lim_{p \rightarrow -\infty} \psi(p)^{1/2}\}.$$

- It exists $\lambda \in N^+ - \{0\}$ such that $\psi(\lambda) = 0$ and ψ is strictly positive without singularity when $x < \lambda$, then $SpN = \lambda + N^-$ and

$$D = \{z; |z| < \lim_{p \rightarrow -\infty} \psi(p)^{1/2}\}.$$

When the eigenvalue of $|0\rangle$ is $\mu \neq 0$, from Proposition 2.1, the point μ , instead of the origin, belongs to SpN and the statements of the propositions must be changed accordingly.

Proposition 2.4 *The nature of the operators (to be creation or annihilation operators) is inverted in the change :*

$$a = a'^{\dagger}, \quad N = -N' - 1, \quad \psi(\rho) = \psi'(-\rho). \quad (8)$$

From this change, we deduce that Proposition 2.3 can be obtained from Proposition 2.2. In the following section, we set up the Bargmann representation when the eigenvectors of the annihilation operator exist and using Proposition 2.4 we obtain the corresponding results when the eigenvectors of the creation operator exist.

3 Bargmann representation

Let $|f\rangle$ be one state of \mathcal{H} :

$$|f\rangle = \sum_{n \in SpN} f_n |n\rangle, \quad \sum_{n \in SpN} |f_n|^2 < \infty \quad (9)$$

with $SpN = \lambda + N^+$ and $\lambda \in N^- + \{0\}$. Following the construction [12], in the Bargmann representation any state $|f\rangle$ of \mathcal{H} is represented by the function of a complex variable z , $f(z) = \langle \bar{z} | f \rangle$:

$$f(z) = \sum_{n \geq 0} z^n f_n \psi(n)!^{-1/2} + \sum_{n < 0} z^n f_n (\psi(n)!)^{1/2}, \quad \sum_{n \geq \lambda} |f_n|^2 < \infty \quad (10)$$

where the variable z belongs to the domain D of definition of the eigenvectors of a .

• Let us summarize the results when the eigenvectors of a exist :

Proposition 3.1 *Let the function ψ characterizing the Deformed Harmonic Oscillator Algebra (1) belong to the first class described in Proposition 2.2. The space \mathcal{S} of the Bargmann representation is constituted with holomorphic functions in $D = \{z; \lim_{p \rightarrow -\infty} \psi(p)^{1/2} < |z| < \lim_{p \rightarrow \infty} \psi(p)^{1/2}\}$, the Laurent expansions of which read (10) with $\lambda = -\infty$. In particular, when $\lim_{p \rightarrow -\infty} \psi(p)^{1/2} = 0$, D is a disk excluding the origin that is a essential singularity point.*

Proposition 3.2 *Let the function ψ characterizing the Deformed Harmonic Oscillator Algebra (1) belong to the second class described in Proposition 2.2. The space \mathcal{S} of the Bargmann representation is the subspace of the space of functions holomorphic in $D = \{z; 0 < |z| < \lim_{p \rightarrow \infty} \psi(p)^{1/2}\}$, the Laurent expansions of which read (10). The functions of \mathcal{S} are of the form $z^\lambda g(z)$ where λ is the lowest bound of the spectrum of N and $g(z)$ is holomorphic in $D + \{0\}$. The origin is a pole of multiplicity lower or equal to $-\lambda$. In particular, when $\lambda = 0$, the functions of \mathcal{S} are holomorphic at the origin.*

In particular, to the basis vectors $|n\rangle, n \in SpN$ correspond the monomials :

$$\langle \bar{z} | n \rangle = \begin{cases} z^n (\psi(n)!)^{-1/2}, & n \geq 0 \\ z^n (\psi(n)!)^{1/2}, & n < 0 \end{cases} \quad (11)$$

- Let us summarize the results when the eigenvectors of a^\dagger exist :

Proposition 3.3 *Let the function ψ characterizing the Deformed Harmonic Oscillator Algebra (1) belong to the first class described in Proposition 2.3. The space \mathcal{S} of the Bargmann representation is constituted with holomorphic functions in $D = \{z; \lim_{p \rightarrow +\infty} \psi(p)^{1/2} < |z| < \lim_{p \rightarrow -\infty} \psi(p)^{1/2}\}$, The functions of \mathcal{S} can be expanded in Laurent series of the form :*

$$f(z) = \sum_{n \geq 0} z^{-n} f_n \psi(n)!^{1/2} + \sum_{n < 0} z^{-n} f_n (\psi(n)!)^{-1/2}, \quad \sum_{n \leq \lambda-1} |f_n|^2 < \infty \quad (12)$$

with $\lambda = +\infty$.

In particular, when $\lim_{p \rightarrow \infty} \psi(p)^{1/2} = 0$, D is a disk excluding the origin that is a essential singularity point.

Proposition 3.4 *Let the function ψ characterizing the Deformed Harmonic Oscillator Algebra (1) belong to the second class described in Proposition 2.3. The space \mathcal{S} of the Bargmann representation is the subspace of the space of functions holomorphic in $D = \{z; 0 < |z| < \lim_{p \rightarrow \infty} \psi(p)^{1/2}\}$ that read (12).*

The functions of \mathcal{S} are of the form $z^{-\lambda+1}g(z)$ where $\lambda - 1$ is the upper bound of the spectrum of N and $g(z)$ is holomorphic in $D + \{0\}$. The origin is a pole of multiplicity lower or equal to $\lambda - 1$. In particular, when $\lambda = 1$, the functions of \mathcal{S} are holomorphic at the origin.

In particular, to the basis vectors $|n\rangle, n \in SpN$ correspond the monomials :

$$\langle \bar{z} | n \rangle = \begin{cases} z^{-n} (\psi(n)!)^{1/2}, & n \geq 0 \\ z^{-n} (\psi(n)!)^{-1/2}, & n < 0 \end{cases} \quad (13)$$

The function of z , $G(\zeta z) = \langle \bar{z} | \zeta \rangle$ corresponds to the coherent state $|\zeta\rangle$. The holomorphic functions belonging to \mathcal{S} have analytical properties

strongly depending on ψ . Their growth on the edge of D is controlled by the growth of $|G(z\bar{z})|^{1/2}$.

A Bargmann representation exists if we can obtain a positive real function $F(x)$ such as

$$\int F(z\bar{z}) | \bar{z} \rangle \langle \bar{z} | dz d\bar{z} = 1 \quad (14)$$

where the integration is extended to the whole complex plane and where $F(|z|^2)$ contains the characteristic function of the domain D of existence of the coherent states. The existence of (14) ensures that the scalar product of the representation takes the form (2).

We easily prove that,

Proposition 3.5 *In a Bargmann representation :*

- either a^\dagger is the multiplication by z , a the operator $z^{-1}\psi(zd/dz)$ and N the operator zd/dz
- or a is the multiplication by z , a^\dagger the operator $z^{-1}\psi(-zd/dz)$ and N the operator $-zd/dz - 1$.

Let us introduce the Mellin transform $\hat{F}(\rho)$ of $F(x)$:

$$\hat{F}(\rho) = \int_0^\infty F(x)x^{\rho-1}dx \quad (15)$$

From (11) and (14), we deduce that $\hat{F}(\rho)$ exists on all the integers belonging to the spectrum of N and verify the following condition :

$$\hat{F}(n+1) = \begin{cases} \psi(n)!, & n \geq 0 \\ (\psi(n)!)^{-1}, & n < 0 \end{cases}, n \in SpN \quad (16)$$

Let us remark that

$$\hat{F}(\rho) \leq \hat{F}(n) + \hat{F}(n+1), \quad n \leq Re\rho < n+1 \quad (17)$$

because $F(x)$ is a positive function. Therefore, the Mellin transform of F exists for any ρ such as $Re\rho \in [\lambda+1, +\infty[$.

Formula (16) is equivalent to

$$\hat{F}(n+1) = \psi(n)\hat{F}(n), \quad \text{with} \quad \hat{F}(1) = 1 \quad (18)$$

which ensures that the operators $a^\dagger = z$, $a = z^{-1}\psi(zd/dz)$ be adjoint on the basis $|n\rangle$. In [2] and [3], we have discussed the interpolation of this equation, the simplest one reads :

Proposition 3.6 *The function ψ characterizing the DHOA defined in (1) and the Mellin transform of the weight function F are related by the following equation :*

$$\hat{F}(\rho + 1) = \psi(\rho)\hat{F}(\rho), \quad \text{with} \quad \hat{F}(1) = 1. \quad (19)$$

when the coherent states are the eigenvectors of the annihilation operator a .

We obtain an analogous proposition when the basis vectors of the Bargmann Hilbert space are the eigenvectors of a^\dagger . In this case, the annihilation operator is the multiplication by z .

Proposition 3.7 *The function ψ characterizing the DHOA defined in (1) and the Mellin transform of the weight function F are related by the following equation :*

$$\hat{F}(-\rho + 1) = \psi(\rho + 1)\hat{F}(-\rho), \quad \text{with} \quad \hat{F}(1) = 1. \quad (20)$$

when the coherent states are the eigenvectors of the creation operator a^\dagger .

When the eigenvalue μ is different from zero, the only change in Equations (19) and (20) is $\psi(\rho) \rightarrow \psi(\mu + \rho)$.

Proposition 3.6 corresponds to functions ψ that are finite and positive on a non upper bounded interval $]\lambda, +\infty[$ with $\lambda \in N^- + \{0\}$, as proved in Proposition 2.2. The Mellin transform of F given by (19) is finite and positive on the same interval. When λ is finite, $\hat{F}(\lambda)$ is infinite. When λ is infinite, $\lim_{\rho \rightarrow -\infty} \frac{\hat{F}(\rho+1)}{\hat{F}(\rho)} < \lim_{\rho \rightarrow +\infty} \frac{\hat{F}(\rho+1)}{\hat{F}(\rho)}$.

Proposition 3.7 corresponds to functions ψ that are finite and positive on a non lower bounded interval $]-\infty, \lambda' + 1[$ with $\lambda' \in N^+$ as proved in Proposition 2.3. The Mellin transform of F given by (20) is finite and positive on the interval $]-\lambda', +\infty[$. When λ' is finite, $\hat{F}(-\lambda')$ is infinite. When λ' is infinite, $\lim_{\rho \rightarrow -\infty} \frac{\hat{F}(-\rho+1)}{\hat{F}(-\rho)} > \lim_{\rho \rightarrow +\infty} \frac{\hat{F}(-\rho+1)}{\hat{F}(-\rho)}$.

These two cases lead to the same conclusions :

Proposition 3.8 *The Mellin transform of the weight function of the Bargmann representation is finite and positive on a non upper bounded interval of the real axis.*

- *When the lower bound of the interval is finite, it is an negative integer or zero and \hat{F} is infinite at this point.*
- *When the interval is not lower bounded, \hat{F} fulfills the following condition :*

$$\lim_{\rho \rightarrow -\infty} \frac{\hat{F}(\rho + 1)}{\hat{F}(\rho)} < \lim_{\rho \rightarrow +\infty} \frac{\hat{F}(\rho + 1)}{\hat{F}(\rho)} \quad (21)$$

Let remark that, while the spectrum of the energy operator N is a non lower or non upper bounded interval in Z , the definition domain of the Mellin transform of the weight function must be a non upper bounded interval of the real axis containing the origin in order that the construction be possible.

4 Deformed harmonic oscillator constructed from a weight function

The purpose of the two following sections is to construct a DHOA assumed to admit a given Bargmann representation. More precisely, we start with a Bargmann Hilbert space \mathcal{S} of functions, holomorphic on a ring D of the complex plane. The scalar product is written on the form (2) in terms of a given function F positive on the interval $]\alpha, \beta[$.

The construction can be summarize as follows : We look for a function ψ , we associate to this function the DHAO defined by (1). Then, remains a consistency condition : we must prove that this algebra admits a Bargmann representation on \mathcal{S} , as defined in section 3.

Once the DHOA is constructed, the spectrum of N is obtained and, eventually, the representation space must be restricted according to Propositions 3.1, 3.2, 3.3, 3.4.

First, in order to define the function ψ by applying (19) or (20), the Mellin transform \hat{F} of the given weight function F must exist. Secondly, in order to define the basis vectors of the representation as the eigenvectors of the annihilation operator a (or a^\dagger), the function \hat{F} must be define on a non upper bounded interval of the real axis, due to the Proposition 3.8.

Let us denote \hat{F} the Mellin transform of the weight function F :

$$\hat{F}(\rho) = \int_{\alpha}^{\beta} F(x) x^{\rho-1} dx \quad (22)$$

From Proposition 3.8, we obtain the following necessary conditions that must satisfy \hat{F} in order that the construction of the DHOA be possible :

Proposition 4.1 *When a Bargmann Hilbert \mathcal{S} space is given, a necessary condition in order that a DHOA admits a representation on \mathcal{S} is that the Mellin transform of the weight function defining the scalar product (2) exists on a non upper bounded interval of the real axis.*

• *When the lower bound of the interval is finite, it must be a negative integer ν . The functions of the representation space read :*

$$f(z) = \sum_{n \geq \nu-1} \frac{z^n f_n}{\hat{F}(n+1)^{\frac{1}{2}}}, \quad \sum_{n \geq \nu-1} |f_n|^2 < \infty \quad (23)$$

• *When the interval is not lower bounded, \hat{F} must fulfill the following condition :*

$$\lim_{\rho \rightarrow -\infty} \frac{\hat{F}(\rho+1)}{\hat{F}(\rho)} < \lim_{\rho \rightarrow +\infty} \frac{\hat{F}(\rho+1)}{\hat{F}(\rho)}. \quad (24)$$

The functions of the representation space read (23) with $\nu = -\infty$.

When the previous conditions are satisfied, we are faced with two possibilities according as we choose that the basis vectors of the Bargmann representation are the eigenstates of the annihilation or of the creation operator :

Proposition 4.2 *When the Mellin transform of the weight function satisfies the necessary conditions of Proposition 4.1, the function :*

$$\psi(\rho) = \frac{\hat{F}(\rho+1)}{\hat{F}(\rho)} \quad (25)$$

put in the relations (1) defines a DHOA that can be represented on the Bargmann Hilbert space spanned with the eigenvectors of the annihilation operator, equipped with the scalar product (2).

In this representation, the creation is the multiplication by z .

Proposition 4.3 *When the Mellin transform of the weight function satisfies the necessary conditions of Proposition 4.1, the function :*

$$\psi(\rho) = \frac{\hat{F}(-\rho + 2)}{\hat{F}(-\rho + 1)} \quad (26)$$

put in the relations (1) defines a DHOA that can be represented on the Bargmann Hilbert space spanned with the eigenvectors of the creation operator, equipped with the scalar product (2).

In this representation, the annihilation is the multiplication by z .

When the eigenvalue μ of the starting state is different from zero, the right hand side of (25) or (26) is unchanged. The left hand side is replaced by $\psi(\mu + \rho)$, the parameter μ of the representation is involved in the data of the weight function.

All the construction is done in terms of the Mellin transform of the weight function. It is worthwhile to note that the reproducing kernel $G(x)$ is always expressed in terms of \hat{F} , by the same simple formula :

$$G(x) = \hat{F}(1) \sum_{n \geq \nu-1} \frac{x^n}{\hat{F}(n+1)} \quad (27)$$

The main point is to prove that the DHOA so constructed admits a representation on the Bargmann Hilbert space \mathcal{S} , that is we have to establish the consistency of the construction.

$$4.1 \quad D = D_{\alpha\beta} \equiv \{z; \quad 0 < \alpha < |z|^2 < \beta < +\infty\}$$

From (22), we obtain the following inequalities :

$$\begin{cases} \hat{F}(\rho) < \alpha^{-\mu} \hat{F}(\rho + \mu) \\ \hat{F}(\rho) < \beta^{\mu} \hat{F}(\rho - \mu) \end{cases} \quad , \quad \forall \mu > 0 \quad (28)$$

that ensure that if $\hat{F}(\rho)$ diverges for one value of ρ , it always diverges and that the Mellin transform of F does not exist on the real axis. In the following we assume that \hat{F} exists. Then the Mellin transform (22) exists for any $\rho \in \mathbb{R}$ and is a strictly positive function. The functions ψ defined by (25) and (26) also are strictly positive functions. Using these functions ψ in (1), we obtain the corresponding DHOAs.

Since the function ψ is strictly positive on R , the spectrum of N is Z .

The construction is achieved if we prove that the coherent states of the so constructed DHOAs exist when $\alpha < |z|^2 < \beta$. We then have to determine the behaviors of the function ψ at $\pm\infty$.

Let us write (22) :

$$\hat{F}(\rho) = \rho^{-1} \beta^\rho \int_{(\frac{\alpha}{\beta})^\rho}^1 F(\beta x^{\frac{1}{\rho}}) dx \quad (29)$$

We easily see that $\hat{F}(\rho) \simeq F(\beta) \rho^{-1} \beta^\rho$ when ρ goes to $+\infty$, if $F(\beta)$ is finite and different from zero. Therefore, the function $\psi(\rho)$ given in (25) (resp. (26)) goes to β when ρ goes to $+\infty$ (resp. $-\infty$).

When $F^{(k)}(\beta) = 0, k = 0, \dots, b$ and when $F^{(b+1)}(\beta)$ is finite and not equal to zero, $\hat{F}(\rho) \simeq (-1)^{b+1} F^{(b+1)}(\beta) \rho^{-(b+2)} \beta^{b+1+\rho}$ when ρ goes to $+\infty$. Then, the function $\psi(\rho)$ given in (25) (resp. (26)) goes to β when ρ goes to $+\infty$ (resp. $-\infty$).

The same reasoning holds for the behavior at $-\infty$, indeed we now write (22) :

$$\hat{F}(\rho) = \rho^{-1} \alpha^\rho \int_1^{(\frac{\beta}{\alpha})^\rho} F(\alpha x^{\frac{1}{\rho}}) dx \quad (30)$$

We get now that the limit at $-\infty$:

When $F(\alpha)$ is finite and different from zero, $\hat{F}(\rho) \simeq F(\alpha) \rho^{-1} \alpha^\rho$.

When $F^{(k)}(\alpha) = 0, k = 0, \dots, a$ and when $F^{(a+1)}(\alpha)$ is finite and different from zero, $\hat{F}(\rho) \simeq (-1)^{a+1} F^{(a+1)}(\alpha) \rho^{-(a+2)} \alpha^{a+1+\rho}$. On these conditions, $\psi(\rho)$ given in (25) (resp. (26)) goes to α when ρ goes to $-\infty$ (resp. $+\infty$).

As $\alpha < \beta$, the necessary condition given in Proposition 4.1 is fulfilled. Using the results of section 3, we prove that the coherent states, eigenstates of a or a^\dagger according to the choice of the characteristic function, are defined when $\alpha < |z|^2 < \beta$. This completes the proof of the consistency of the reconstruction.

We obtain the following sufficient conditions in order that the construction be consistent :

Proposition 4.4 *Let $F(x)$ be a function defined on the interval $] \alpha, \beta[$ such as :*

- *the Mellin transform of F exists,*
- *$F(\alpha)$ is finite and different from zero or $F^{(a+1)}(\alpha)$ is finite and different from zero when $F^{(k)}(\alpha) = 0$, for $k = 0, \dots, a$,*

- $F(\beta)$ is finite and different from zero or $F^{(b+1)}(\beta)$ is finite and different from zero when $F^{(l)}(\beta) = 0$, for $l = 0, \dots, b$.

One can construct two Deformed Harmonic Oscillator Algebras that admits a representation on a Hilbert space constituted by functions, holomorphic in the ring $D_{\alpha\beta} = \{z; \quad \alpha^{\frac{1}{2}} < |z| < \beta^{\frac{1}{2}}\}$ and equipped with the scalar product :

$$(g, f) = \int F(z\bar{z}) f(z) \overline{g(z)} \theta(z\bar{z} - \alpha) \theta(\beta - z\bar{z}) dz d\bar{z} \quad (31)$$

- One of this DHOA corresponds to the characteristic function (25) and in the Bargmann representation, its creation operator is the multiplication by z .
- The second DHOA corresponds to the characteristic function (26) and in the Bargmann representation, its annihilation operator is the multiplication by z .

If the edge conditions are not fulfilled the reconstruction may exist but we have to establish in each specific cases that the limits at infinities are the expected ones.

Finally, let us remark that

$$\alpha \int_{\alpha}^{\beta} F(x) x^{\rho-1} dx \leq \int_{\alpha}^{\beta} F(x) x^{\rho} dx \leq \beta \int_{\alpha}^{\beta} F(x) x^{\rho-1} dx \quad (32)$$

Obviously, these inequalities hold when $\alpha = 0$ or β infinite. We obtain :

$$\alpha \leq \psi(\rho) \leq \beta \quad (33)$$

This constitutes a necessary condition that the function ψ must fulfill in order that the DHOA have a Bargmann representation :

Proposition 4.5 *Let the function ψ characterizing the deformed harmonic oscillator algebra (1) be strictly positive and such that its limits at infinities are finite and verify $\psi(-\infty) < \psi(+\infty)$ (resp. $\psi(-\infty) > \psi(+\infty)$). The coherent states, eigenstates of the annihilation (resp. creation) operator, exist in the ring of the complex plane*

$$\begin{aligned} D &= \{z; \quad \psi(-\infty)^{1/2} < |z| < \psi(+\infty)^{1/2}\} \\ (\text{resp. } D &= \{z; \quad \psi(+\infty)^{1/2} < |z| < \psi(-\infty)^{1/2}\}). \end{aligned} \quad (34)$$

The DHOA can be represented on a space of functions holomorphic in D only if ψ takes all its values between the two limiting values :

$$\begin{aligned} \psi(-\infty) &< \psi(\rho) < \psi(+\infty) \\ (\text{resp. } \psi(+\infty) &< \psi(\rho) < \psi(-\infty)). \end{aligned} \quad (35)$$

4.2 $D = D_{0\beta} \equiv \{z; \quad |z|^2 < \beta < +\infty\}$

The second inequality of (28) still holds. Let us denote by ν the value such as the integration (22) is divergent when $\rho \leq \nu$ and convergent when $\rho > \nu$. When $\nu = -\infty$ the Mellin transform exists for any ρ , while when $\nu = +\infty$ it never exists. As $\hat{F}(\nu)$ is infinite while $\hat{F}(\nu + 1)$ is finite, $\psi(\nu) = 0$ (resp. $\psi(1 - \nu) = 0$) defined by (25) (resp. (26) vanishes. The spectrum of N is $\nu + N^+$ (resp. $2 - \nu + N^-$) when ν is finite and Z when $\nu = -\infty$. From Proposition 4.1, ν must belong to $N^- + \{0\}$.

The expressions of the coherent states contain one or two infinite summations according to ν is finite or not.

- In the case where $\nu = -\infty$, $SpN = Z$, the limit at $-\infty$ must be done in each specific case and we treat an example in the subsection (8.1). This limit is positive or zero since $\hat{F}(\rho)$ is positive, it must be zero in order that the domain of existence of the coherent states be consistent with the definition of the given Bargmann Hilbert space.

Let us summarize this result :

Proposition 4.6 *Let $F(x)$ be a function defined on the interval $]0, \beta[$ such as :*

- *the Mellin transform of F exists on the whole real axis,*
- *$F(\beta)$ is finite and different from zero or $F^{(b+1)}(\beta)$ is finite and different from zero when $F^{(l)}(\beta) = 0$, for $l = 0, \dots, b$,*
- *and $\lim_{\rho \rightarrow -\infty} \frac{\hat{F}(\rho+1)}{\hat{F}(\rho)} = 0$.*

One can construct two deformed harmonic oscillator algebras, corresponding to characteristic functions given in (25) and (26), that admit a representation on a Hilbert space constituted by functions, holomorphic in the ring $D = \{z; \quad 0 < |z|^2 < \beta^{\frac{1}{2}}\}$ and equipped with the scalar product :

$$(g, f) = \int F(z\bar{z}) f(z) \overline{g(z)} \theta(\beta - z\bar{z}) dz d\bar{z} \quad (36)$$

- In the case where ν is finite, it remains only one infinite summation in (7), this summation is convergent if $|z|^2 < \lim_{\rho \rightarrow +\infty} \psi(\rho)$. As β is finite, this limit is obtained by the same reasoning as in the previous case. In this case, as the spectrum of N is lower or upper bounded according as we choose ψ defined by (25) or by (26), the Laurent expansions of the functions belonging to the representation space contain terms in z^n with $n \geq \nu$ as results from Propositions 3.2 and 3.4. We then have :

Proposition 4.7 *Let $F(x)$ be a function defined on the interval $]0, \beta < +\infty[$ such as :*

- *the Mellin transform of F only exists when $\rho > \nu$, ν finite and belonging to $N^- + \{0\}$.*
- *$F(\beta)$ is finite and different from zero or $F^{(b+1)}(\beta)$ is finite and different from zero when $F^{(l)}(\beta) = 0$, for $l = 0, \dots, b$.*

One can construct two deformed harmonic oscillator algebras, corresponding to characteristic functions given in (25) and (26), that admit a representation on a Hilbert space equipped with a scalar product (36) and constituted with functions holomorphic in the ring $D = \{z; \quad 0 < |z| < \beta^{\frac{1}{2}}\}$ restricted by the condition that the origin be a pole of multiplicity lower or equal to ν .

In particular when $\nu = 0$, the functions of the representation space are holomorphic in a disk including the origin.

When β is infinite, the consistency of the demonstration must be done in each case, one example is given in subsection 8.2.

4.3 $D = D_{\alpha\infty} \equiv \{z; \quad 0 < \alpha |z|^2\}$

In this case, the integration (22) can diverge for $\nu < \rho$ and converge for $\rho \leq \nu$. If ν is finite, $\psi(\nu)$ defined by equation (25) and $\psi(1 - \nu)$ defined by equation (26) is infinite. This corresponds to a case where ψ has a singularity at a finite distance, not considered in this paper. If $\nu = +\infty$, (22) always converges and the spectrum of N is Z , the consistency must be verified in each case.

In this section, we have obtained consistency sufficient conditions to construct of a DHOA from its Bargmann representation when D is a true ring or a true disk in the complex plane, an example will be given in the subsection 5.3.

5 Examples of construction

In this section, we give four examples of construction of DHOA when its Bargmann representation is given, namely F and the domain D of existence of the coherent states are given. In the first two examples D is the whole complex plane, then the sufficient conditions of consistency of the previous section do not apply. In the last two ones it is a ring, one of them illustrates

the results of the Propositions 4.5 and 4.6, while in the last one the sufficient conditions of the previous section are not fulfilled.

In the following, the proofs are given for DHOA resulting from characteristic functions given in (25) and for which the coherent states are the eigenvectors of the annihilation operator. Obviously, the same can be developed when ψ is given by (26), leading to DHOA for which the coherent states are the eigenvectors of the creation operator, we just state the results.

5.1 $D = C - \{0\}$ and $F(x) = \exp(-\sigma(\ln x)^{2n})$

We assume that σ is a positive real number and that n is a positive integer. As the domain of existence of the coherent states is the whole complex plane, the sufficient condition of the previous section are not fulfilled. This example is an illustration of a case where the existence of the DHOA is established though the characteristic function ψ is not obtained on an explicit form. The Mellin transform of $F(x)$ (15) reads :

$$\hat{F}(\rho) = \int_{-\infty}^{+\infty} e^{-\sigma t^{2n} + \rho t} dt \quad (37)$$

As σ is positive and n is a positive integer, we see that $\hat{F}(\rho)$ exists and is strictly positive for all ρ . The function ψ given by (25) is then a strictly positive function and the spectrum of N is \mathbb{Z} .

The reconstruction of the deformed algebra will be achieved if we prove that the coherent states resulting of the function ψ such obtained are defined in the whole complex plane as assumed. We thus have to study the behavior of ψ at infinities.

Let us assume that ρ is positive. We write (37):

$$\hat{F}(\rho) = \int_{-\infty}^{+\infty} e^{-\sigma(t+v)^{2n} + \rho(t+v)} dt \quad (38)$$

where we choose

$$v = \left(\frac{\rho}{\sigma(2n-1)} \right)^{\frac{1}{2n-1}} \quad (39)$$

in order that the term under the exponential does not contain linear term in t and (38) reads :

$$\hat{F}(\rho) = e^{-2(n-1)\sigma \frac{1}{2n-1} \left(\frac{\rho}{2n-1}\right)^{\frac{2n}{2n-1}}} \int_{-\infty}^{+\infty} dt e^{-\sigma \left(\sum_{p \geq 2} C_p^{2n} t^p v^{2n-p}\right)} \quad (40)$$

After a change of variable $t = uv^{1-n}$, (40) can be written :

$$\hat{F}(\rho) = e^{-2(n-1)\sigma \frac{1}{2n-1} \left(\frac{\rho}{2n-1}\right)^{\frac{2n}{2n-1}}} \int_{-\infty}^{+\infty} du v^{1-n} e^{-\sigma n(2n-1)u^2} e^{-\sigma \left(\sum_{p \geq 3} C_p^{2n} u^p v^{n(2-p)}\right)} \quad (41)$$

When $\rho \rightarrow +\infty$, the integral goes to zero like v^{1-n} and using (9), we have :

$$\psi(\rho) \simeq e^{\frac{4n(n-1)}{(2n-1)^2} \left(\frac{\rho}{\sigma(2n-1)}\right)^{\frac{1}{2n-1}}} \quad (42)$$

We therefore get :

$$\lim_{\rho \rightarrow +\infty} \psi(\rho) = +\infty \quad (43)$$

From (37), we deduce that

$$\hat{F}(-\rho) = -\hat{F}(\rho) \quad (44)$$

so that

$$\psi(-\rho) = \frac{\hat{F}(-\rho+1)}{\hat{F}(-\rho)} = \frac{\hat{F}(\rho-1)}{\hat{F}(\rho)} = \frac{1}{\psi(\rho-1)} \quad (45)$$

Thus when $\rho \rightarrow -\infty$, we get that

$$\lim_{\rho \rightarrow -\infty} \psi(\rho) = 0. \quad (46)$$

The domain of existence of the coherent states is the complex plane and the consistency of the reconstruction is established.

As $SpN = Z$, the representation space is the space of the functions holomorphic in the complex plane without the origin that is a essential singularity point. Similarly, to Equation (26) corresponds another DHOA.

Proposition 5.1 *One can construct two Deformed Harmonic Oscillator Algebras that can be represented on the Bargmann Hilbert space of functions*

holomorphic in the complex plane without the origin equipped with the scalar product :

$$(g, f) = \int \exp(-\sigma(\log z \bar{z})^{2n}) \overline{g(z)} f(z) dz d\bar{z}$$

- The characteristic function ψ involved in (1) is :

$$\psi(\rho) = \frac{\int_{-\infty}^{+\infty} \exp(-\sigma t^{2n} + (\rho + 1)t) dt}{\int_{-\infty}^{+\infty} \exp(-\sigma t^{2n} + \rho t) dt}$$

and the spectrum of N is \mathbb{Z} and the coherent states are the eigenvectors of a .

- The characteristic function ψ involved in (1) is :

$$\psi(\rho) = \frac{\int_{-\infty}^{+\infty} \exp(-\sigma t^{2n} + (2 - \rho)t) dt}{\int_{-\infty}^{+\infty} \exp(-\sigma t^{2n} + (1 - \rho)t) dt}$$

and the spectrum of N is \mathbb{Z} and the coherent states are the eigenvectors of a^\dagger .

In the next subsection, we give another example where the results of the previous section do not apply and in which the explicit calculation of the characteristic function ψ can be performed. The main interest of the next example is to be a generalization of the usual harmonic oscillator algebra.

5.2 $D = C - \{0\}$ and $F(x) = \exp(-x^{\frac{k}{m}})$

We assume that $\frac{k}{m}$ is put on an irreducible form and that it is positive. When $\frac{k}{m} = 1$, F is the weight function of the Bargmann representation of the usual harmonic oscillator [12].

The Mellin transform of F (15) reads :

$$\hat{F}(\rho) = \int_0^{+\infty} e^{-x^{\frac{k}{m}}} x^{\rho-1} dx \quad (47)$$

After a change of variable $u = x^{\frac{k}{m}}$, it reads

$$\hat{F}(\rho) = \frac{m}{k} \int_0^{+\infty} e^{-u} u^{\rho \frac{m}{k} - 1} du = \frac{m}{k} \Gamma(\rho \frac{m}{k}) \quad (48)$$

and the function ψ characterizing the DHOA and resulting from (25) is :

$$\psi(\rho) = \frac{\Gamma(\frac{m}{k}(\rho + 1))}{\Gamma(\frac{m}{k}\rho)} \quad (49)$$

From this explicit expression of ψ , we deduce that this function is strictly positive on the positive axis and vanishes at the origin. The spectrum of N is N^+ .

Using the asymptotic behavior of $\Gamma(z)$ for large values of $|z|$, we get :

$$\psi(\rho) \simeq \left(\frac{m}{k}\rho\right)^{\frac{m}{k}} \quad (50)$$

Thus $\lim_{\rho \rightarrow +\infty} \psi(\rho)$ is infinite and the coherent states, as assumed, are defined in the whole C . As $SpN = N^+$, the functions of the representation space are holomorphic in C , including the origin. A similar construction can be performed with the characteristic function (26).

Proposition 5.2 *One can construct two Deformed Harmonic Oscillator Algebra that can be represented on the Bargmann Hilbert space of functions holomorphic in the whole complex plane equipped with the scalar product :*

$$(g, f) = \int \exp(-(z\bar{z})^{\frac{k}{m}}) \overline{g(z)} f(z) dz d\bar{z}, \quad \frac{k}{m} > 0$$

- The characteristic function ψ is :

$$\psi(\rho) = \frac{\Gamma(\frac{m}{k}(\rho + 1))}{\Gamma(\frac{m}{k}\rho)}$$

The spectrum of N is N^+ and the coherent states are the eigenvectors of a .

- The characteristic function ψ is :

$$\psi(\rho) = \frac{\Gamma(\frac{m}{k}(2 - \rho))}{\Gamma(\frac{m}{k}(1 - \rho))}$$

The spectrum of N is $N^- + \{0\}$ and the coherent states are the eigenvectors of a^\dagger .

When $\frac{k}{m} = 1$, the function ψ resulting of Equation (49) is the characteristic function of the usual harmonic oscillator in a fixed representation, namely $a^\dagger a = N$.

We end this subsection by comparing the Bargmann representation considered in this subsection with the Bargmann representation of the usual harmonic oscillator.

In this subsection, the scalar product (2) is defined in the space \mathcal{S} of holomorphic functions of one complex variable and reads :

$$(g, f) = \int d\zeta d\bar{\zeta} e^{-\zeta \bar{\zeta} \frac{k}{m}} f(\zeta) \overline{g(\zeta)}, f, g \in \mathcal{S} \quad (51)$$

Denoting $\zeta = \chi e^{i\tau}$, it can be written :

$$(g, f) = m^{-1} \int_0^{2\pi m} \frac{d\tau}{2\pi} \int_0^{+\infty} d\chi^2 e^{-\chi^2 \frac{2k}{m}} f(\chi e^{i\tau}) \overline{g(\chi e^{i\tau})} \quad (52)$$

The scalar product for the usual Bargmann representation reads :

$$(g_B, f_B) = \int dz d\bar{z} e^{-z\bar{z}} f_B(z) \overline{g_B(z)}, f_B, g_B \in \mathcal{S}_B \quad (53)$$

It takes the form :

$$(g_B, f_B) = k^{-1} \int_0^{2\pi k} \frac{d\theta}{2\pi} \int_0^{+\infty} d\rho^2 e^{-\rho^2} f_B(\rho e^{i\theta}) \overline{g_B(\rho e^{i\theta})} \quad (54)$$

Let us change $z = \zeta \frac{k}{m}$, we see that $0 \leq \tau < 2\pi m$ and that (54) reads :

$$(g_B, f_B) = \frac{k}{m^2} \int_0^{2\pi m} \frac{d\tau}{2\pi} \int_0^{+\infty} d\chi^2 e^{-\chi^2 \frac{2k}{m}} \zeta^{\frac{k}{m}-1} f_B(\zeta \frac{k}{m}) \overline{\zeta^{\frac{k}{m}-1} g_B(\zeta \frac{k}{m})} \quad (55)$$

The scalar products written in (55) and in (52) are the same but they are not defined on the same space of functions :

Indeed let us write $f_B(z) = \sum_{l \leq 0} f_l z^l$, the functions $f(\zeta) \equiv \zeta^{\frac{k}{m}-1} f_B(\zeta \frac{k}{m})$ belong to \mathcal{S} iff $f_l = 0$ when $l \neq nm - 1$, n being a strictly positive integer. The functions f such obtained belong to \mathcal{S} but do not cover the whole space for they read :

$$f(z) = \sum_{n=1}^{+\infty} f_{nm-1} z^{kn-1} \quad (56)$$

Proposition 5.3 *Let us consider the two Bargmann Hilbert spaces on which are represented the usual harmonic oscillator algebra and the DHOA considered in this subsection. When, by a change of variables, their scalar products are written on the same form (52), the functions belonging to the intersection of these two spaces are of the form (56).*

5.3 $D = D_{\alpha\beta}$ and $F(x) = x^\sigma$

We start with a Bargmann representation such as the coherent states are defined on a ring of the complex plane $0 \leq \alpha \leq \rho \leq \beta < +\infty$. This subsection illustrates the previous section with an example where we obtain an explicit expression for the characteristic function ψ .

The Mellin transform of the weight function reads :

$$\hat{F}(\rho) = \int_{\alpha}^{\beta} x^{\sigma+\rho-1} dx \quad (57)$$

First, we see that this integration is finite for any ρ and any σ when $\alpha \neq 0$ and for $\rho > -\sigma$ when $\alpha = 0$.

The resulting function ψ , defined by (25), takes the form :

$$\psi(\rho) = \frac{\sigma + \rho}{\sigma + \rho + 1} \frac{\beta^{\sigma+\rho+1} - \alpha^{\sigma+\rho+1}}{\beta^{\sigma+\rho} - \alpha^{\sigma+\rho}} \quad (58)$$

We now must look for the domain of existence of the coherent states in order to verify the consistency of this construction.

- When $\alpha \neq 0$, the function ψ is always positive and the spectrum of N is Z . It is easy to find that the function $\psi(\rho)$ goes to α or β when $\rho \rightarrow -\infty$ or $+\infty$. This implies that the coherent states are defined for $\alpha \leq \rho^2 \leq \beta$, as expected. As $SpN = Z$, no restrictions appear on the Laurent expansions of the holomorphic functions of the representation space. The same construction can be done starting with the characteristic function (26)

Proposition 5.4 *One can construct two Deformed Harmonic Oscillator Algebras that can be represented on the space of functions holomorphic in $D_{\alpha\beta} = \{z; \quad 0 < \alpha < |z|^2 < \beta\}$ equipped with the scalar product :*

$$(g, f) = \int_{0 < \alpha < |z|^2 < \beta} (z\bar{z})^\sigma \overline{g(z)} f(z) dz d\bar{z}, \quad \forall \sigma.$$

The characteristic functions are $\psi(\rho)$ or $\psi(1-\rho)$ expressed in (58), SpN is Z in both cases and the coherent states are the eigenvectors of the annihilation or of the creation operator.

- When $\alpha = 0$, the characteristic function resulting from (25) is :

$$\psi(\rho) = \frac{\sigma + \rho}{\sigma + \rho + 1} \beta \quad (59)$$

From Proposition 3.8, we deduce that the construction is only possible when $\sigma \in N^+$. Then $SpN = -\sigma + N^+$. The coherent states are defined in $D = \{z; |z|^2 < \beta\}$ as expected and the origin is a pole of multiplicity lower than σ for the functions of the representation space. Let us summarize this result and that obtained starting with Equation (26) :

Proposition 5.5 *One can construct two Deformed Harmonic Oscillator Algebras that can be represented on the space of functions of the form $z^{-\sigma} f_0(z)$ where $f_0(z)$ is holomorphic in the whole disk $D_{0\beta} = \{z; |z|^2 < \beta\}$ equipped with the scalar product :*

$$(g, f) = \int_{|z|^2 < \beta} (z\bar{z})^\sigma f(z) \overline{g(z)} dz d\bar{z} \quad (60)$$

provided that σ be zero or a positive integer. Their characteristic functions are $\psi(\rho)$ or $\psi(1-\rho)$ written in Equation (59). The spectrum of N is $-\sigma + N^+$ or $\sigma + N^- + \{0\}$ and the coherent states are the eigenvectors of the annihilation or of the creation operator.

5.4 $D = D_{0\beta}$ and $F(x) = x^\sigma(\beta - x)^\eta$

This example does not fulfill the general conditions of Section 4 for the derivatives of the weight function on the edge β of D is zero or infinite when η is not a positive integer. The Mellin transform of F exists when $\rho + \sigma > 0$ and $\eta + 1 > 0$ and can be calculated in terms of the B function [13] :

$$\begin{aligned} \hat{F}(\rho) &= \int_0^\beta (\beta - x)^\eta x^{\sigma+\rho-1} dx \\ &= \beta^{\eta+\sigma+\rho} B(\rho + \sigma, \eta + 1) \\ &= \beta^{\eta+\sigma+\rho} \frac{\Gamma(\rho+\sigma)\Gamma(\eta+1)}{\Gamma(\rho+\sigma+\eta+1)} \end{aligned} \quad (61)$$

The expression of $\hat{F}(\rho)$ put in Equation (25) leads to :

$$\psi(\rho) = \frac{\rho + \sigma}{\rho + \sigma + \eta + 1} \beta \quad (62)$$

When $\eta = 0$, one recovers the function (59). The reasoning and the results are similar to those of Proposition 5.5 :

Proposition 5.6 *One can construct two Deformed Harmonic Oscillator Algebras that can be represented on the space of functions of the form $z^{-\sigma} f_0(z)$ where $f_0(z)$ is holomorphic in the whole disk $D = \{z; |z|^2 < \beta\}$ equipped with the scalar product :*

$$(g, f) = \int_{|z|^2 < \beta} (\beta - z\bar{z})^\eta (z\bar{z})^\sigma f(z) \overline{g(z)} dz d\bar{z} \quad (63)$$

provided that σ be zero or a positive integer. Their characteristic functions are $\psi(\rho)$ or $\psi(1-\rho)$ written in Equation (62). The spectrum of N is $-\sigma + N^+$ or $\sigma + N^- + \{0\}$ and the coherent states are the eigenvectors of the annihilation or of the creation operator.

5.5 $D = D_{\alpha\beta}$ and $F(x) = \exp(x - \beta)^{-1}$

In this example, the general conditions of the section 4 are not fulfilled too, for all the derivatives of $F(x)$ vanish on the edge β of D . To simplify, we give the proofs for $\alpha = 0$.

The Mellin transform of $F(x)$ are defined for $\rho > 0$. Integrating by parts, we get :

$$\hat{F}(\rho) = \frac{1}{\rho} \int_0^\beta \exp\left(\frac{1}{x - \beta}\right) \frac{x^\rho}{(\beta - x)^2} dx \quad (64)$$

Expanding $(\beta - x)^{-2}$, we obtain the relation :

$$\rho = \beta^{-2} \sum_{n \geq 0} \frac{n}{\beta^n} \frac{\hat{F}(\rho + n + 1)}{\hat{F}(\rho)} \quad (65)$$

Now, when $\rho \rightarrow \infty$, $\frac{\hat{F}(\rho+1)}{\hat{F}(\rho)}$ necessarily goes to the limit $\beta_0 \leq \beta$ according Proposition 4.5. This implies :

$$\lim_{\rho \rightarrow \infty} \frac{\hat{F}(\rho + n + 1)}{\hat{F}(\rho)} = \beta_0^n \quad (66)$$

This result is used to calculate the right hand side of Equation (65) when ρ goes to ∞ . As the limit of the left hand side is infinite, we obtain a contradiction unless $\beta_0 \neq \beta$. As $\hat{F}(0)$ is infinite, $\psi(0) = 0$ and the coherent states are defined in $D_{0\beta}$. The consistency conditions are satisfied. Let us state the result for arbitrary α :

Proposition 5.7 *One can construct two Deformed Harmonic Oscillator Algebras that can be represented on the space of functions holomorphic in $D_{\alpha\beta}$ equipped with the scalar product :*

$$(g, f) = \int_{\alpha < |z|^2 < \beta} \exp(z\bar{z} - \beta) f(z) \overline{g(z)} dz d\bar{z} \quad (67)$$

The spectrum of N is N^+ or $N^- + \{0\}$ when $\alpha = 0$ and Z when $\alpha \neq 0$ and the coherent states are the eigenvectors

- *of the annihilation when the characteristic function is*

$$\psi(\rho) = \frac{\int_{\alpha}^{\beta} \exp(x - \beta)^{-1} x^{\rho} dx}{\int_{\alpha}^{\beta} \exp(x - \beta)^{-1} x^{\rho-1} dx} \quad (68)$$

- *or of the creation operator when the characteristic function is $\psi(1 - \rho)$.*

From these examples, we conjecture that the propositions of Section 4 can be largely extended.

6 Conclusion

Giving a ring D in the complex plane and a positive function F that characterize a functional Bargmann Hilbert space \mathcal{S} , we have discussed the conditions under which exists a DHOA that admits a Bargmann representation in \mathcal{S} , as defined in 1.2. We have obtained conditions on the weight function in order that solutions exist :

- in Proposition 4.1, we give necessary conditions,
- in Propositions 4.4, 4.5, 4.6, 4.7, we give sufficient conditions, when D is not the whole complex plane.

When one DHOA solution exists, another DHOA exists, if the annihilation operator of the first one possesses eigenvectors generating the representation space, the same holds for the creation operator of the second one.

Finally, we have developed some examples that does not fulfill the sufficient conditions, in particular, when D is the whole complex plane. We have obtained deformations of usual Harmonic Oscillator Algebra through deformations of its Bargmann representation.

References

- [1] M.Irac-Astaud and G.Rideau *Proceedings of the Symposium in honor of Jiri Patera and Pavel Winternitz for their 60th birthday, Algebraic Methods and Theoretical Physics, January 9-11, 1997, Centre de recherches mathématiques, Université de Montréal.*
- [2] M.Irac-Astaud and G.Rideau *Reviews in Mathematical Physics, to appear*
- [3] M.Irac-Astaud and G.Rideau *Czech.J.Phys. Vol 47, 11, 1147 (1997), 6th Colloquium on Quantum Group and Integrable systems, Prague, 19-21 June 1997*
- [4] L.C.Biedenharn *J.Phys.A: Math. Gene. 22 (1989) L873*
- [5] A.J.Mac Farlane *J.Phys.A: Math. Gene. 22 (1989) 4581*
- [6] M.Irac-Astaud and G.Rideau *Proc.Third Int.Wigner Symposium (Oxford, 1993), to appear*
- [7] M.Irac-Astaud and G.Rideau "On the existence of quantum bihamiltonian systems : the harmonic oscillator case", preprint PAR-LPTM 92, *Lett. Math. Phys. 29 (1993) 197, Theor.Math. Phys. 99(1994) 658*
- [8] R.W. Gray and C.A. Nelson *J.Phys. A : Math. Gen.23 , (1990) L 945*
- [9] M.Chaichian and A.Demichev " *Introduction to Quantum Groups*" *World Scientific (1996)*
- [10] C.Quesne and N.Vansteenkiste *Helv.Phys.Acta 69 (1996) 141*, and many references therein.

- [11] P.Kosinski, M.Mazewski and P.Maslanka *Czech.J.Phys.*47, (1997), 41,
5th Colloquium on quantum groups and integrable systems, Prague, 20-22 June 1996
- [12] V.Bargmann *Communications on pure and applied mathematics*,
14,(1961), 187
- [13] I.S.Gradshteyn and I.M.Ryzhik "*Table of integrals series and products*",
Academic Press (1965)